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Generalized Vector Variational Inequalities over Countable Product of Sets

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Abstract. In this paper, we consider vector variational inequalities with set-valued mappings over countable product sets in a real Banach space setting. By employing concepts of relative pseudomonotonicity, we establish several existence results for generalized vector variational inequalities and for systems of generalized vector variational inequalities. These results strengthen previous existence results which were based on the usual monotonicity type assumptions.

Key words: Countable product of sets, Existence results, Relative pseudomonotonicity, Set-valued mappings, Vector variational inequalities

1. Introduction

A great number of economic problems concern the existence of competitive equilibria in exchange economies with infinitely many commodities. More precisely these problems arise in an intertemporal economy with an infinite number of time periods, a differentiation of commodities, uncertainty with an infinite number of states, for example in Nash equilibrium strategy in a large sequential game with individual uncertainties. In this environment of uncertainty the appropriate model for the space of commodities is an infinite-dimensional vector space (Aliprantis et al., 1989; Kreps, 1979). Moreover most number of equilibrium type problems have a decomposable structure, namely, they can be formulated as vector variational inequalities over Cartesian product sets (see, e.g., Yuan, 1998; Yang and Goh, 1997; and the references therein). Most existence results for such problems are based on the known fixed point techniques, which require either the feasible set (otherwise, the corresponding subset associated to a coercivity condition) be compact in the strong topology or the cost mapping possess certain continuity type properties with respect to the weak topology (see, e.g., Yuan, 1998; Ansari and Yao, 1999, 2000). Usually, to essentially weaken these assumptions one makes use of the Ky Fan Lemma, Ky Fan (1961) together with certain monotonicity type properties regardless of the decomposable structure of VI (see, e.g., Hadjisavvas and Schaible, 1998; Oettli and Schläger, 1998). However, it was noticed by Bianchi (1993) that infinite-dimensional extensions of the concepts of M- and P-mappings are not sufficient to apply the Ky Fan Lemma for deriving existence results.

In this paper, we develop some other approach, which is based on the invariance of the solution sets of decomposable equilibrium and variational inequality problems with respect to certain linear transformations. This property enables one to extend the usual (generalized) monotonicity properties. Rosen (1965) introduced such an extension of strict monotonicity to establish the uniqueness of solutions for non-cooperative games with scalar payoffs. Being based on the same property, Konnov (2001) introduced new (generalized) monotonicity concepts, which are adjusted to a decomposable structure of the initial problem, and proved new existence results for scalar variational inequalities in a Banach space. These new relative monotonicity concepts can be regarded as intermediate ones between the usual monotonicity and order monotonicity ones. In Allevi et al. (2001), we extended the results from Konnov (2001) to vector variational inequalities with set-valued mappings over the Cartesian product of a finite number of sets. Now we consider the countable case, thus extending the results from Konnov (2001) and Allevi et al. (2001). In the infinite case, there are additional difficulties in deriving existence results for such problems.

In this paper, we give an approach to handle variational inequalities over the Cartesian product of a countable number of sets, which are based on new concepts of generalized monotonicity. Namely, we establish existence results for generalized vector variational inequalities and for systems of generalized vector variational inequalities over countable product of sets in a real Banach space by employing new relative (pseudo)monotonicity concepts for set-valued mappings.

2. Problem formulations and basic facts

Let I be an infinite countable set of indices. For each $s \in I$, let E_s be a real Banach space and U_s be a nonempty subset of E_s. Set

$$U = \prod_{s \in I} U_s. \tag{1}$$

Let F be a real Banach space space with a partial order induced by a convex, closed and solid cone C. Denote by $\mathbf{R}^\infty$ the set of all infinite sequences with elements from $\mathbf{R}$, namely, $\mathbf{R}^\infty = (\mu_s \in \mathbf{R} \mid s \in I)$. Also, set

$$\mathbf{R}^\infty_> = \{\mu \in \mathbf{R}^\infty \mid \mu_i > 0, i \in I\}.$$

For each $s \in I$, let $G_s : U \to 2^{L(E_s,F)}$ be a mapping so that if we set

$$G = (G_s \mid s \in I), \tag{2}$$

then $G : U \to 2^{L(E,F)}$, where

$$E = \prod_{s \in I} E_s.$$

The *generalized vector variational inequality problem* (GVVI) is to find an element $u^* = (u_s^*)_{s \in I} \in U$ such that

$$G(u^*)(u - u^*) = \sum_{s \in I} G_s(u^*)(u_s - u_s^*) \not\subseteq -\text{int } C \quad \forall u_s \in U_s, s \in I. \tag{3}$$

Together with GVVI (3) we shall consider its dual formulation, which is to find an element $u^* = (u_s^*)_{s \in I} \in U$ such that

$$\forall s \in I, \forall u_s \in U_s : \quad \sum_{s \in I} G_s(u)(u_s - u_s^*) \not\subseteq -\text{int } C. \tag{4}$$

We denote by U^* and U^d the solution sets of problems (3) and (4), respectively.

Next, we shall also consider the *system of generalized vector variational inequalities* (SGVVI for short), which is to find an element $u^* = (u_s^*)_{s \in I} \in U$ such that

$$G_s(u^*)(u_s - u_s^*) \not\subseteq -\text{int } C \quad \forall u_s \in U_s, s \in I. \tag{5}$$

In the scalar case, SGVVI (5) is clearly equivalent to GVVI (3), however, in the general case, SGVVI has a number of its own applications and it is also investigated intensively (see, e.g., Yang and Goh, 1997; Ansari and Yao, 2000). We denote by U^s the solution set of SGVVI (5).

Throughout this paper, we shall apply the weak topology in E, the strong topology in F, and the strong operator topology in $L(E, F)$. We now give some relationships between SGVVI and GVVI under the known continuity and monotonicity type properties of set-valued mappings.

DEFINITION 1. The mapping $G : U \to 2^{L(E,F)}$, defined by (2), is said to be

(a) *u-hemicontinuous* (Definition 2.2, Konnov and Yao, 1997) if for any $u, v \in U$ and $\lambda \in [0, 1]$, the mapping $\lambda \to G(u + \lambda z)z$ with $z = v - u$ is upper semicontinuous at 0^+;
(b) *pseudo* (w, P)*-monotone* (Definition 1, Allevi et al. 2001) if for all $u, v \in U$, we have

$$G_s(v)(u_s - v_s) \not\subseteq -\text{int } C \; \forall s \in I \implies \sum_{s \in I} G_s(u)(u_s - v_s) \not\subseteq -\text{int } C.$$

It should be noted that in the scalar case where $F = \mathbf{R}$, Definition 1 (b) is a modification of Definition 2.2 in Konnov (1999) (see also Definition 1, Bianchi, 1993).

LEMMA 1. *GVVI (3) implies SGVVI (5).*

The proof follows immediately from (1) and (2).

LEMMA 2. *If the mapping $G : U \to 2^{L(E,F)}$, defined by (2), is pseudo (w, P)-monotone, then $U^s \subseteq U^d$.*

The proof is straightforward.

LEMMA 3. *Suppose that the set U, defined by (1), is convex and that the mapping $G : U \to 2^{L(E,F)}$, defined by (2), is u-hemicontinuous. Then $U^d \subseteq U^*$.*

The proof is the same as in (Lemma 2.1, Konnov and Yao, 1997).
Combining Lemmas 1–3, we obtain the following immediately.

PROPOSITION 1. *Suppose that the set U, defined by (1), is convex and that the mapping $G : U \to 2^{L(E,F)}$, defined by (2), is u-hemicontinuous and pseudo (w, P)-monotone. Then GVVI (3) is equivalent to SGVVI (5).*

The following well-known Ky Fan Lemma (Lemma 1, Ky Fan, 1961; see also Yuan, 1998 p. 6) will play a crucial role in deriving existence results for GVVIs and SGVVIs.

PROPOSITION 2. *Let X and Y be non-empty sets in a topological vector space E and $Q : X \to 2^Y$ be such that*

(i) *for each $x \in X$, $Q(x)$ is closed in Y;*
(ii) *for each finite subset $\{x^1, \cdots, x^n\}$ of X, its convex hull is contained in the corresponding union $\bigcup_{i=1}^n Q(x^i)$;*
(iii) *there exists a point $\tilde{x} \in X$ such that $Q(\tilde{x})$ is compact.*
Then

$$\bigcap_{x \in X} Q(x) \neq \emptyset.$$

3. Relative monotonicity type properties

In Konnov (2001), new monotonicity type concepts for single-valued mappings, which extend the usual ones, were proposed. Now, we extend these concepts to multi-valued vector mappings over countable product of sets.

DEFINITION 2. The mapping $G : U \to 2^{L(E,F)}$, defined by (2), is said to be

(a) *relatively monotone* if there exist vectors $\alpha, \beta \in \mathbf{R}^{\infty}_{>}$ such that for all $u, v \in U$, we have
$$\sum_{s \in I} [\alpha_s G_s(u) - \beta_s G_s(v)] (u_s - v_s) \subseteq C;$$

(b) *relatively w-pseudomonotone* if there exist vectors $\alpha, \beta \in \mathbf{R}^{\infty}_{>}$ such that for all $u, v \in U$, we have

$$\sum_{s\in I} \beta_s G_s(v)(u_s - v_s) \not\subseteq -\text{int}\, C \Longrightarrow \sum_{s\in I} \alpha_s G_s(u)(u_s - v_s) \not\subseteq -\text{int}\, C.$$

In what follows, we reserve the symbols α and β for parameters associated to relative (pseudo) monotonicity of G. It is clear that relative monotonicity implies relative w-pseudomonotonicity, but the reverse assertion is not true in general. Note that relative monotonicity type properties obviously extend the usual monotonicity type ones (see, e.g., Konnov and Yao, 1997; Hadjisavvas and Schaible, 1998; Oettli and Schläger, 1998). In the special case where $F = \mathbf{R}$, $C = \mathbf{R}^{+}$, we obtain relative (pseudo)monotonicity concepts for the cost mapping $G : U \to 2^{E^*}$ of the scalar generalized variational inequality. Here and below E^* denotes the topological conjugate space of E.

We now consider a parametric form of GVVI. Fix an element $\gamma \in \mathbf{R}^{\infty}_{>}$ and consider the mapping $G^{(\gamma)} : U \to 2^{L(E,F)}$ which is defined by

$$G^{(\gamma)} = (\gamma_s G_s \mid s \in I).$$

Replacing in (3) and (4) the mapping G by $G^{(\gamma)}$, we obtain the following parametric GVVIs: Find an element $u^* \in U$ such that

$$G^{(\gamma)}(u^*)(u - u^*) = \sum_{s\in I} \gamma_s G_s(u^*)(u_s - u^*_s) \not\subseteq -\text{int}\, C \quad \forall u_s \in U_s, s \in I; \tag{6}$$

and find an element $u^* \in U$ such that

$$\forall s \in I, \forall u_s \in U_s : \quad \sum_{s\in I} G^{(\gamma)}_s(u)(u_s - u^*_s) \not\subseteq -\text{int}\, C. \tag{7}$$

We denote by U^*_γ and U^d_γ the solution sets of problems (6) and (7), respectively. We now adjust the assertions of Lemmas 1 and 3 to the parametric GVVIs.

LEMMA 4.

(i) *GVVI* (6) *implies SGVVI* (5).
(ii) *If the set U, defined by* (1), *is convex and that the mapping $G^{(\gamma)} : U \to 2^{L(E,F)}$, defined by* (2), *is u-hemicontinuous, then $U^d_\gamma \subseteq U^*_\gamma$.*

Proof. Part (i) follows immediately from Lemma 1. Part (ii) follows from Lemma 3. □

4. Existence results

We now establish existence results for generalized vector variational inequalities and for systems of generalized vector variational inequalities. First we recall the definition of a completely continuous mapping.

DEFINITION 3. (e.g., Diestel and Uhl, 1970) A mapping is said to be *completely continuous* if it maps each weakly convergent sequence into a strongly convergent sequence.

THEOREM 1. *Let U be convex and weakly compact. Suppose that G is relatively w-pseudomonotone and that the operators G, $G^{(\alpha)}$, and $G^{(\beta)} : U \to 2^{L(E,F)}$ have nonempty values on U. Suppose also that $G^{(\alpha)}$ is u-hemicontinuous and has compact values and, in addition, each element of $G^{(\alpha)}(u)$ is completely continuous at every point $u \in U$. Then SGVVI (5) is solvable.*

Proof. Define set-valued mappings $A, B : U \to 2^U$ by

$$B(v) = \left\{ u \in U \mid \sum_{s\in I} \beta_s G_s(u)(v_s - u_s) \not\subseteq -\text{int}\, C \right\}$$

and

$$A(v) = \left\{ u \in U \mid \sum_{s\in I} \alpha_s G_s(v)(v_s - u_s) \not\subseteq -\text{int}\, C \right\}.$$

We divide the proof into the following three steps.

(i) $\bigcap_{v\in U} \overline{B(v)}^{w} \neq \emptyset$. Let z be in the convex hull of any finite subset $\{v^1, \dots, v^n\}$ of U. Then $z = \sum_{j=1}^{n} \mu_j v^j$ for some $\mu_j \geqslant 0$, $j = 1, \dots, n$; $\sum_{j=1}^{n} \mu_j = 1$. If $z \notin \cup_{j=1}^{n} B(v^j)$, then for all $g_s \in G_s(z)$, $s \in I$, we have

$$\sum_{s\in I} \beta_s g_s(v_s^j - z_s) \in -\text{int}\, C \quad \forall j = 1, \dots, n.$$

Since $-\text{int}\, C$ is convex, we obtain

$$\sum_{j=1}^{n} \mu_j \left(\sum_{s\in I} \beta_s g_s(v_s^j - z_s) \right) \in -\text{int}\, C.$$

It follows that

$$\begin{aligned} 0 &= \sum_{s\in I} \beta_s g_s(z_s - z_s) \\ &= \sum_{s\in I} \beta_s g_s \left(\sum_{j=1}^{n} \mu_j v_s^j - \sum_{j=1}^{n} \mu_j z_s \right) \\ &= \sum_{j=1}^{n} \mu_j \left(\sum_{s\in I} \beta_s g_s(v_s^j - z_s) \right) \in -\text{int} C, \end{aligned}$$

a contradiction. Therefore, the mapping $\overline{B}^w : U \to 2^U$ satisfies all the assumptions of Proposition 2 and we get

$$\bigcap_{v\in U} \overline{B(v)}^w \neq \emptyset.$$

(ii) $\bigcap_{v\in U} A(v) \neq \emptyset$. From relative w-pseudomonotonicity of G it follows that $B(v) \subseteq A(v)$, but, for each $v \in U$, $A(v)$ is weakly closed. In fact, let $\{u^\theta\}$ be a net in $A(v)$ such that u^θ converges weakly to $\bar{u} \in U$. Then, for each θ, there exist elements $g_s^\theta \in G_s^{(\alpha)}(v)$, $s \in I$, such that

$$\sum_{s\in I} g_s^\theta(v_s - u_s^\theta) \notin -\mathrm{int}C.$$

Since $G^{(\alpha)}(v)$ is compact, without loss of generality we can suppose that $g^\theta \to \bar{g} \in G^{(\alpha)}(v)$. It follows that

$$g^\theta(u^\theta) \to \bar{g}(\bar{u})$$

and

$$\sum_{s\in I} g_s^\theta(v_s - u_s^\theta) \to \sum_{s\in I} \bar{g}_s(v_s - \bar{u}_s) \notin -\mathrm{int}C.$$

We conclude that $\bar{u} \in A(v)$, i.e. $A(v)$ is weakly closed. Therefore, $\overline{B(v)}^w \subseteq A(v)$ and (i) now implies (ii).

(iii) $U^s \neq \emptyset$. From (ii) it follows that $U_\alpha^d \neq \emptyset$. Applying now Lemma 4 yields $U^s \neq \emptyset$, as desired.

Thus, SGVVI (5) is solvable. □

By employing the corresponding coercivity condition, we obtain existence results on unbounded sets.

COROLLARY 1. *Let U be convex and closed and let there exist a weakly compact subset V of E and a point $\tilde{v} \in V \cap U$ such that*

$$\sum_{s\in I} \beta_s G_s(u)(\tilde{v}_s - u_s) \subseteq -\mathrm{int}\, C \quad \textit{for all} \quad u \in U \backslash V. \tag{8}$$

Suppose that G is relatively w-pseudomonotone and that the operators G, $G^{(\alpha)}$, and $G^{(\beta)} : U \to 2^{L(E,F)}$ have nonempty values on U. Suppose also that $G^{(\alpha)}$ is u-hemicontinuous and has compact values and, in addition, each element of $G^{(\alpha)}(u)$ is completely continuous at every point $u \in U$. Then SGVVI (5) is solvable.

Proof. In this case it suffices to follow the proof of Theorem 1 and observe that $B(\tilde{v}) \subseteq V$ under the above assumptions. Indeed, it follows that $\overline{B(\tilde{v})}^w$ is weakly compact, hence the assertion of Step (i) will be true due to Proposition 2 as well. □

Observe that the assertions of Theorem 1 and Corollary 1 can be viewed as extensions of the results of Theorem 1 and Corollary 2 in Konnov (2001) for the multi-valued vector case. Combining the above results with Proposition 1, we obtain existence results of solutions for GVVI.

THEOREM 2. *Suppose that G is relatively w-pseudomonotone and pseudo* (w, P)-*monotone and that the operators* G, $G^{(\alpha)}$, *and* $G^{(\beta)} : U \to 2^{L(E,F)}$ *have nonempty values on* U. *Suppose also that* G *and* $G^{(\alpha)}$ *are u-hemicontinuous,* $G^{(\alpha)}$ *has compact values, and, in addition, each element of* $G^{(\alpha)}(u)$ *is completely continuous at every point* $u \in U$. *Suppose that* U *is convex and that at least one of the following assumptions holds*:

(i) U *is weakly compact*;
(ii) U *is closed and there exist a weakly compact subset* V *of* E *and a point* $\tilde{v} \in V \cap U$ *such that (8) holds.*

Then GVVI (3) is solvable.

The proof follows from Theorem 1, Corollary 1, and Proposition 1.

5. Scalarization of GVVIs

In this section, we introduce some other relative (pseudo)monotonicity concepts which admit equivalent scalar ones and present existence results for generalized vector variational inequalities by way of solving an appropriate generalized variational inequalities. Such a scalarization approach for generalized vector variational inequality problems with general feasible sets was suggested in Konnov and Yao (1997).

Given an element $z \in F^*$ and a mapping $G : U \to 2^{L(E,F)}$, we define the mapping $G_z : U \to 2^{E^*}$ by

$$(G_z(u), v) = (z, (G(u)v))$$

for $u \in U$ and $v \in E$. Clearly, if the mapping $G : U \to 2^{L(E,F)}$ is defined by (2), then

$$G_z = (G_{s,z} \mid s \in I),$$

where $G_{s,z} : U \to 2^{E_s^*}$ is defined as follows

$$(G_{s,z}(u), v_s) = (z, (G_s(u)v_s))$$

for $u \in U$ and $v_s \in E_s$. Also, set

$$H(z) = \{f \in F \mid (z, f) \geqslant 0\}.$$

We now introduce relative (pseudo)monotonicity concepts which are different from those in Definition 2.

DEFINITION 4. Let z be an element in $F^*\backslash\{\mathbf{0}\}$. The mapping $G : U \to 2^{L(E,F)}$, defined by (2), is said to be

(i) *relatively monotone with respect to* z if there exist vectors $\alpha, \beta \in \mathbf{R}^{\infty}_{>}$ such that for all $u, v \in U$, we have
$$\left[G^{(\alpha)}(u) - G^{(\beta)}(v)\right](u - v) \subseteq H(z);$$

(ii) *relatively* w*-pseudomonotone with respect to* z if there exist vectors $\alpha, \beta \in R^{\infty}_{>}$ such that for all $u, v \in U$, we have
$$G^{(\beta)}(v)(u - v) \cap H(z) \neq \emptyset \Longrightarrow G^{(\alpha)}(u)(u - v) \cap H(z) \neq \emptyset.$$

It is clear that relative monotonicity with respect to z implies relative w-pseudomonotonicity with respect to z, but the reverse assertion is not true in general. Also, if $C \subseteq H(z)$, then relative monotonicity implies relative monotonicity with respect to z. These new relative (pseudo)monotonicity concepts for the mapping G are closely related with those from Definition 2 for G_z as the following proposition states.

PROPOSITION 3. *The mapping* $G : U \to 2^{L(E,F)}$*, defined by (2), is relatively monotone with respect to* z *(respectively, relatively* w*-pseudomonotone with respect to* z*) for some* z *in* $F^*\backslash\{\mathbf{0}\}$ *if and only if the mapping* $G_z : U \to 2^{E^*}$ *is relatively monotone (respectively, relatively* w*-pseudomonotone).*

Proof. First, it is clear that, for all $u', u'' \in U$, the relation
$$(p_z^{(\alpha)} - q_z^{(\beta)}, u'' - u') \geqslant 0$$
for all $q_z^{(\beta)} \in G_z^{(\beta)}(u')$ and $p_z^{(\alpha)} \in G_z^{(\alpha)}(u'')$ is equivalent to
$$(p^{(\alpha)} - q^{(\beta)})(u'' - u') \in H(z)$$
for all $q^{(\beta)} \in G^{(\beta)}(u')$ and $p^{(\alpha)} \in G^{(\alpha)}(u'')$. Next, for any $u', u'' \in U$, suppose that
$$(q_z^{(\beta)}, u'' - u') \geqslant 0$$
for some $q_z^{(\beta)} \in G_z^{(\beta)}(u')$. Then $(z, (q^{(\beta)}(u'' - u'))) \geqslant 0$ for some $q^{(\beta)} \in G^{(\beta)}(u')$ and $q^{(\beta)}(u'' - u') \in H(z)$. If $G^{(\beta)}$ is relatively w-pseudomonotone with respect to z, then we must have $p^{(\alpha)}(u'' - u') \in H(z)$ for some $p^{(\alpha)} \in G^{(\alpha)}(u'')$. Hence, for some $p_z^{(\alpha)} \in G_z^{(\alpha)}(u'')$,
$$(p_z^{(\alpha)}, u'' - u') \geqslant 0,$$
i.e. G_z is relatively w-pseudomonotone. Conversely, let G_z be relatively w-pseudomonotone. Then, for any $u', u'' \in U$, the relation $q^{(\beta)}(u'' - u') \in H(z)$ for some $q^{(\beta)} \in G^{(\beta)}(u')$ is equivalent to $(z, q^{(\beta)}(u'' - u')) \geqslant 0$, hence it follows that $(q_z^{(\beta)}, u'' - u') \geqslant 0$ for some $q_z^{(\beta)} \in G_z^{(\beta)}(u')$. By relative w-pseudomonotonicity of

G_z, we have $(p_z^{(\alpha)}, u'' - u') \geqslant 0$ for some $p_z^{(\alpha)} \in G_z^{(\alpha)}(u'')$, i.e., $p^{(\alpha)}(u'' - u') \in H(z)$ for some $p^{(\alpha)} \in G^{(\alpha)}(u'')$. It means that $G^{(\beta)}$ is relatively w-pseudomonotone with respect to z. □

Thus, one can verify such a relative (pseudo)monotonicity of cost mappings of generalized vector variational inequalities with the help of the corresponding properties of cost mappings of scalar generalized variational inequalities. We now turn to existence results for generalized vector variational inequalities under relative (pseudo)monotonicity with respect to a vector z. We will denote by C^* the conjugate cone to C, i.e.,

$$C^* = \{z \in F^* \mid (z, f) \geqslant 0 \quad \forall f \in C\}.$$

THEOREM 3. *Let U be convex and weakly compact. Suppose that G is relatively w-pseudomonotone with respect to z for some z in $C^*\backslash\{\mathbf{0}\}$ and that the operators G, $G^{(\alpha)}$, and $G^{(\beta)} : U \to 2^{L(E,F)}$ have nonempty values on U. Suppose also that $G^{(\alpha)}$ is u-hemicontinuous and has compact values and, in addition, each element of $G^{(\alpha)}(u)$ is completely continuous at every point $u \in U$. Then, there exists a solution to GVVI (3).*

Proof. Since $z \in C^*\backslash\{\mathbf{0}\}$, the mapping G_z is relatively w-pseudomonotone due to Proposition 3. Besides, under the assumptions of the present theorem we see that the operators G_z, $G_z^{(\alpha)}$, and $G_z^{(\beta)} : U \to 2^{E^*}$ have nonempty values on U, $G_z^{(\alpha)}$ is u-hemicontinuous and has compact values and each element of $G_z^{(\alpha)}(u)$ is completely continuous at every point $u \in U$. Therefore, in the special case where $F = \mathbf{R}$, $C = \mathbf{R}^+$, Theorem 1 guarantees the existence of a solution $\bar{u} \in U$ for GVI_z, i.e.,

$$\forall v \in U,\ \exists \bar{g}_z \in G_z(\bar{u}) : (\bar{g}_z, v - \bar{u}) \geqslant 0. \tag{9}$$

Fix an arbitrary point $v \in U$. Then, by the above, there exists $\bar{g} \in G(\bar{u})$ such that

$$(z, (\bar{g}(v - \bar{u}))) \geqslant 0,$$

hence, $\bar{g}(v - \bar{u}) \notin -\mathrm{int}H(z)$. Since $z \in C^*$, $-\mathrm{int}H(z) \supseteq -\mathrm{int}C$, so that

$$\bar{g}(v - \bar{u}) \notin -\mathrm{int}C.$$

Therefore, $\bar{u}$ is a solution to GVVI (3). The proof is complete. □

Notice that the above existence result does not requre for G to be pseudo (w, P)-monotone. In addition, we give a similar result for GVVI with an unbounded domain with the help of the following coercivity condition.

DEFINITION 5. Let z be an element in $F^*\backslash\{\mathbf{0}\}$. The mapping $G : U \to 2^{L(E,F)}$, defined by (2), is said to be *weakly v-coercive* if there exist $u_0 \in U$ and $z \in C^*\backslash\{\mathbf{0}\}$ such that

$$\inf_{g \in G_z(u)} (g, u - u_0) \to \infty \quad \text{as } u \in U,\ \|u\|_E \to \infty.$$

THEOREM 4. *Let U be a convex closed subset of a reflexive Banach space E. Suppose that G is relatively w-pseudomonotone and weakly v-coercive with respect to z for some z in $C^*\backslash\{\mathbf{0}\}$ and that the operators G, $G^{(\alpha)}$, and $G^{(\beta)} : U \to 2^{L(E,F)}$ have nonempty values on U. Suppose also that $G^{(\alpha)}$ is u-hemicontinuous and has compact values and, in addition, each element of $G^{(\alpha)}(u)$ is completely continuous at every point $u \in U$. Then, there exists a solution to GVVI* (3).

Proof. From the proof of Theorem 3 we conclude that it is sufficient to prove that there exists a solution to problem (9). Let B_r denote the closed ball (under the norm) of E with center at $\mathbf{0}$ and radius r. Set $V_r = U \cap B_r$. Again, in the special case where $F = \mathbf{R}$, $C = \mathbf{R}^+$, Theorem 1 guarantees the existence of a solution $u_r \in V_r$ for the following problem:

$$\exists \bar{g}_z \in G_z(u_r) : (\bar{g}_z, v - u_r) \geqslant 0 \quad \forall v \in V_r.$$

Choose $r \geqslant \|u_0\|_E$, where u_0 satisfies the weak v-coercivity of G. Then, for some $g_z' \in G_z(u_r)$,

$$(g_z', u_0 - u_r) \geqslant 0. \tag{10}$$

We observe that the set $\{u_r \mid r > 0\}$ must be bounded. In fact, otherwise we can choose r large enough so that the weak v-coercivity of G yields

$$(g_z,\ u_0 - u_r) < 0 \quad \forall\, g_z \in G_z(u_r),$$

which contradicts to (10). Therefore, there exists r such that $\|u_r\|_E < r$. Now, for each $u \in U$, we can choose $\varepsilon > 0$ small enough such that $u_r + \varepsilon(u - u_r) \in V_r$. Then,

$$(\bar{g}_z, u_r + \varepsilon(u - u_r) - u_r) \geqslant 0$$

for some $\bar{g}_z \in G_z(u_r)$. Dividing ε on both sides of the above inequality, we obtain

$$(\bar{g}_z, u - u_r) \geqslant 0,$$

which shows that u_r is a solution of problem (9) and the result follows. □

In order to illustrate the usefulness of the relative monotonicity concepts we give now the simplest example. We restrict ourselves with the scalar case since the vector case can be considered similarly. Namely, let us consider the case where $E = l_2$, $F = \mathbf{R}$, and $G : E \to E$ is a linear operator of the form

$$G(u) = Iu + Au,$$

where I is the identity map in l_2, A is the infinite matrix with zero entries with the exception of the submatrix

$$\tilde{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} 2/3 & 1 \\ 3 & 1 \end{pmatrix}.$$

Taking the point $u = (1, -1, 0, \dots, 0, \dots)^T \in l_2$, we have $\langle G(u), u\rangle = -1/3$, i.e. G is not monotone, and the usual theory does not guarantee the existence of solutions for VI with the cost mapping G. At the same time, using the representation $l_2 = \prod_{s=1}^{\infty} \mathbf{R}$, we see that the mapping $G^{(\alpha)}$ with $\alpha = (3, 1, 1, \dots, 1, \dots)$ is monotone (even coercive), hence G is relatively monotone and we can obtain the existence results directly from Theorem 1 or Corollary 1.

Thus, we see that even simplest perturbations of coericive diagonal operators are serious drawbacks for applying the theory of (vector) VIs with usual (generalized) monotone operators, whereas the new concepts remain still useful.

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